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First Fundamental Theorem for Covariants of Classical Groups

D. A. Shmelkin[1]

Independent Moscow University, Moscow, Russia

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Let $U(G)$ be a maximal unipotent subgroup of one of the classical groups $G = GL(V), O(V), Sp(V)$. Let W be a direct sum of copies of V and its dual V^*. For the natural action $U(G) : W$, we describe a minimal system of homogeneous generators for the algebra of $U(G)$-invariant regular functions on W. For $G = O(V), Sp(V)$, this result is connected with a construction for the irreducible

1. INTRODUCTION

Let V be a finite-dimensional vector space over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $H \subseteq GL(V)$ be an algebraic subgroup. For any $l \in \mathbf{Z}_+$, we denote by lV the direct sum of l copies of V; similarly, we define mV^* for any $m \in \mathbf{Z}_+$. Consider the natural action of H on $W = lV \oplus mV^*$ and assume that the algebra $\mathbf{k}[W]^H$ of the H-invariant regular functions on W is finitely generated for any l, m. Then the First Fundamental Theorem of Invariant Theory of H refers to a description of a minimal system of homogeneous generators of $\mathbf{k}[W]^H$ for all l, m.

Since the 19th century such a description has been known for *classical* H, i.e., for H being one of the groups $GL(V)$, $SL(V)$, $O(V)$, $SO(V)$, or $Sp(V)$. In modern mathematics these results are of interest after the famous book [We] of H. Weyl. However, only a few new examples of groups (H, V) such that the first fundamental theorem can be proven have been found since: $(SL_2, S^3\mathbf{k}^3)$, $(G_2, \mathbf{k}^7)$, and $(Spin_7, \mathbf{k}^8)$ (see [Sch87], [Sch88]).

Now let G be one of the groups $GL(V), O(V), Sp(V)$; let $U(G)$ be a maximal unipotent subgroup of G. It is well known (e.g., [PV, Theorem 3.13]) that the algebra $\mathbf{k}[W]^{U(G)}$ is finitely generated. The invariants of $U(G)$ are linear combinations of highest weight vectors of

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irreducible factors for the G-module $\mathbf{k}[W]$. So the $U(G)$-invariants are the G-covariants and the First Fundamental Theorem for covariants of G means for the invariants of $H = U(G)$.

In this paper we prove the first fundamental theorem for covariants for each of the above classical G. Note that for $G = Sp(V), O(V)$, V and V^* are isomorphic as G-modules, hence, $U(G)$-modules. Therefore we may assume $m = 0$ in these cases. Some particular cases of our result are known. In [Br85] the algebras of covariants are described in the case when these are polynomial algebras, i.e., for $l, m \leqslant 2$ (assuming $m = 0$ for $O(V), Sp(V)$). Also the case $G = GL(V), m = 0$ is not complicated (see 3.1) and seems to be well known.

A function $f \in \mathbf{k}[W]$ is called *multilinear antisymmetric*, if f belongs to $\bigwedge^p V^* \subseteq \bigotimes^p V^* \subseteq \mathbf{k}[pV]$, for some $p \leqslant l$ or to a similar subspace in $\mathbf{k}[qV^*]$.

THEOREM 1.1. *The algebra $\mathbf{k}[W]^{U(G)}$ is generated by the subalgebra $\mathbf{k}[W]^G$ and multilinear antisymmetric invariants. Moreover, a set $\mathcal{M}$ described in Section 2 is a minimal system of homogeneous generators of $\mathbf{k}[W]^{U(G)}$.*

In this paper we give several proofs of the above theorem. More precisely, for each of the classical groups we give two independent proofs. A uniform proof for all three groups is presented in Section 6. This proof is based on results of Howe [Ho] concerning the structure of the isotypic components of G-module $\mathbf{k}[W]$ and was the first proof we found. However at first we present other proofs of the above theorem.

For $GL(V)$, we describe in Section 3 the syzygies of the system $\mathcal{M}$ from 1.1, in some particular cases (see 3.3). This allows us to produce an independent proof of 1.1 for these particular cases (see 3.4). Furthermore, in Section 4 we generalize some results of Schwarz from [Sch87] (see 4.3) and obtain 1.1 in the whole generality as a corollary.

After a preliminary version of this paper was written, Schwarz observed that for $O(V)$ and $Sp(V)$, Theorem 1.1 is connected with a construction of Weyl of the irreducible representations of these groups in terms of tensor algebra [We]. This construction is well known (see, e.g., [FH]) but the connection between both topics is not evident. Modulo some generalities, Theorem 1.1 follows from Weyl's Theorem 5.2. This is presented in Section 5.

2. MINIMAL SYSTEM OF GENERATORS

We describe a minimal system $\mathcal{M}$ of homogeneous generators of $\mathbf{k}[W]^{U(G)}$ in a coordinate form. Set $n = \dim V$, choose a basis of V, and denote by $\bar{V}$ the corresponding $n\times l$-matrix of coordinates on lV. Similarly, denote by $\overline{V^*}$ the $m\times n$-matrix of coordinates on mV^*, in the dual basis of V^*. A minor of order k of a matrix is said to be *left* if it involves the first k columns. Analogously, we call it *lower* if it involves the last k rows.

(A) Let $G=GL(V)$ and define $U(GL)=U(GL(V))$ to be the subgroup of the upper triangular matrices with units on the diagonal, in the above basis. Then $\mathcal{M}$ is:

- the matrix elements of the product $\overline{V^*}\,\bar{V}$,
- the lower minor determinants of order k of $\bar{V}$, $k=1, \ldots, \min\{l, n\}$,
- the left minor determinants of order p of $\overline{V^*}$, $p=1, \ldots, \min\{m, n\}$.

Let $T(GL)$ be the diagonal matrices in the above basis. Then $T(GL)$ is a maximal torus of G normalizing $U(GL)$. Denote by ε_i the weight of the ith basis vector with respect to $T(GL)$, $i=1, \ldots, n$. Then $\varphi_k=\varepsilon_1+\cdots+\varepsilon_k$, $k=1, \ldots, n$ are the fundamental weights of $T(GL)$ with respect to $U(GL)$. The torus $T(GL)$ acts on $\mathbf{k}[W]^{U(GL)}$ and the elements of $\mathcal{M}$ are weight vectors of $T(GL)$. The set of their degrees and weights is (for $l, m\geqslant n$):

$$(2, 0), (1, \varphi_1), (2, \varphi_2), \ldots, (n, \varphi_n),$$
$$(1, \varphi_{n-1}-\varphi_n), (2, \varphi_{n-2}-\varphi_n), \ldots, (n-1, \varphi_1-\varphi_n), (n, -\varphi_n).$$

Furthermore, let Q be a bilinear symmetric (antisymmetric) form having in the above basis a matrix with ± 1 on the secondary diagonal and with zero entries outside it. Define $G=O(V)$ ($G=Sp(V)$) to be the stabilizer of this form. Then $U(G)=G\cap U(GL)$ is a maximal unipotent subgroup in G. Moreover, set $T(O)=T(GL)\cap SO(V)$, $T(Sp)=T(GL)\cap Sp(V)$. Then $T(G)$ is a maximal torus of G of rank $r=[\frac{n}{2}]$. Denote by $\varphi_1, \ldots, \varphi_r$ the fundamental weights of $T(G)$ with respect to $U(G)$ (see, e.g., [OV]). For $x\in W=lV$, denote by v_i the projection of x on the ith V-factor, $i=1, \ldots, l$.

(B) Let $n=2r+1$, $G=O(V)$. Then $\mathcal{M}$ is:

- $Q(v_i, v_j)$, $1\leqslant i\leqslant j\leqslant l$,
- the lower minor determinants of order k of $\bar{V}$, $k=1, \ldots, \min\{l, n\}$.

The set of degrees and weights of the above generators is (for $l\geqslant n$):

$$(2, 0), (1, \varphi_1), \ldots, (r-1, \varphi_{r-1}), (r, 2\varphi_r), (r+1, 2\varphi_r), \ldots, (n-1, \varphi_1), (n, 0).$$

(C) Let $G=Sp(V)$. Then $\mathcal{M}$ is:

- $Q(v_i, v_j)$, $1\leqslant i<j\leqslant l$,
- the lower minor determinants of order k of $\bar{V}$, $k=1, \ldots, \min\{l, r\}$.

The set of degrees and weights of the above generators is (for $l \geqslant r$):

$$(2, 0), (1, \varphi_1), (2, \varphi_2), \ldots, (r, \varphi_r).$$

Note that the lower minor determinants of order k of $\bar{V}$ with $k > r$ are $U(Sp)$-invariant, too. However by 4.3, these can be expressed in the above generators.

(D) Let $n = 2r$, $G = O(V)$. Then $\mathscr{M}$ is:

- $Q(v_i, v_j)$, $1 \leqslant i \leqslant j \leqslant l$,
- the lower minor determinants of order k of $\bar{V}$, $k = 1, \ldots, \min\{l, n\}$,
- for $l \geqslant r$, the minor determinants of order r, involving the rth row and the last $r-1$ rows of $\bar{V}$.

The set of degrees and weights of the above generators is (for $l \geqslant n$):

$$(2, 0), (1, \varphi_1), \ldots, (r-2, \varphi_{r-2}), (r-1, \varphi_{r-1}+\varphi_r), (r, 2\varphi_{r-1}), (r, 2\varphi_r),$$
$$(r+1, \varphi_{r-1}+\varphi_r), \ldots, (n-1, \varphi_1), (n, 0).$$

3. SYZYGIES OF THE MINOR DETERMINANTS

In this section we give a proof of Theorem 1.1 for $G = GL(V)$ and $l + m \leqslant n$ based on a study of the syzygies (i.e., algebraic relations) of the system $\mathscr{M}$.

Set $U = U(GL)$ and denote by W_U the spectrum of $\mathbf{k}[W]^U$. Moreover, denote by $\pi_{U,W}$ the quotient map $\pi_{U,W}: W \to W_U$ corresponding to the inclusion $\mathbf{k}[W]^U \subseteq \mathbf{k}[W]$.

For any $p, l \in \mathbf{N}$, $1 \leqslant p \leqslant l$, set $L = \mathbf{k}^l \oplus \bigwedge^2 \mathbf{k}^l \oplus \cdots \oplus \bigwedge^p \mathbf{k}^l$. Let $\mathscr{F}_{p,l}$ denote the set of all $(q_1, q_2, \ldots, q_p) \in L$ such that the i-vector q_i is decomposable and $Ann(q_{i-1}) \subseteq Ann(q_i)$, $i = 2, \ldots, p$, where $Ann(q) = \{x \in V \mid q \wedge x = 0\}$.

The subset $\mathscr{F}_{p,l}$ is not closed in L. Indeed, assume $(q_1, \ldots, q_p) \in \mathscr{F}_{p,l}$ is such that $q_2 \neq 0$. Then for any $t \in \mathbf{k}^*$ the collection $(tq_1, q_2, \ldots, q_p)$ also belongs to $\mathscr{F}_{p,l}$. But the limit $(0, q_2, \ldots, q_p)$ of such collections does not belong to $\mathscr{F}_{p,l}$. Denote by $\overline{\mathscr{F}_{p,l}}$ the Zariski closure of $\mathscr{F}_{p,l}$.

Note that the subset $\mathscr{F}_{p,l}$ is stable under the natural action of the group GL_l on L so that GL_l acts on $\mathscr{F}_{p,l}$ and $\overline{\mathscr{F}_{p,l}}$.

Now we prove Theorem 1.1 for $G = GL(V)$, $m = 0$:

THEOREM 3.1. *For $W = lV$, consider the rows $u_1, \ldots, u_n$ of the matrix $\bar{V}$ as coordinates of some vectors in $\mathbf{k}^l$. For $p = \min\{l, n\}$ the map $W \to \overline{\mathscr{F}_{p,l}} \subseteq L$ taking a n-tuple of vectors to the element with coordinates*

$$(u_n, u_{n-1} \wedge u_n, \ldots, u_{n-p+1} \wedge u_{n-p} \wedge \cdots \wedge u_n)$$

is the GL_L-equivariant quotient map $\pi_{U,W}$ and its image is $\mathscr{F}_{p,l}$.

Proof. The map described in the statement is clearly GL_l-equivariant and its image is all of $\mathscr{F}_{p,l}$. Therefore we only need to prove that the Plücker coordinates of the antisymmetric forms $u_n, u_{n-1} \wedge u_n, \ldots, u_{n-p+1} \wedge \cdots \wedge u_n$ generate $\mathbf{k}[W]^U$. But these are just the lower minor determinants of $\bar{V}$ so that Theorem 3.1 implies Theorem 1.1.

Let the maximal unipotent subgroup $U' \subseteq GL_l$ consist of all the upper triangular matrices with units on the diagonal, in the chosen basis of $\mathbf{k}^l$. It is well known (see, e.g., [Kr, 3.7]) that $\mathbf{k}[W]^{U \times U'}$ is generated by the left lower minor determinants of $\bar{V}$. Therefore the algebra A generated by all the lower minor determinants contains $\mathbf{k}[W]^{U \times U'}$. In other words, $A^{U'} = (\mathbf{k}[W]^U)^{U'}$. Since A is GL_l-stable, we obtain $A = \mathbf{k}[W]^U$. ∎

By the above theorem, the syzygies of the lower minor determinants are the generators of the ideal in $\mathbf{k}[L]$ vanishing on $\mathscr{F}_{p,l}$. It is well known that these are the famous Plücker relations. For any $s \leqslant p$, let $M_{i_1, \ldots, i_s}$ be the tensor of the Plücker coordinates of $u_{n-s+1} \wedge \cdots \wedge u_n$ in the standard basis of $\bigwedge^s \mathbf{k}^l$. In other words, M is antisymmetric and for $i_1 < i_2 < \cdots < i_s$ $M_{i_1, \ldots, i_s}$ is the determinant of the lower minor involving the columns number $i_1, \ldots, i_s$ of $\bar{V}$. Using an elegant explicit form of the Plücker relations from [FH, 15.53], we get:

COROLLARY 3.2. *The ideal of syzygies for the lower minor determinants is generated by the following Plücker relations: for any $t \leqslant s \leqslant p$, any s-tuple $\{i_1; i_2; \ldots; i_s\}$, and any t-tuple $\{j_1; j_2; \ldots; j_t\}$,*

$$M_{i_1, \ldots, i_s} M_{j_1, \ldots, j_t} = \sum_{a=1}^{s} M_{i_1, \ldots, i_{a-1}, j_1, i_{a+1}, \ldots, i_s} M_{i_a, j_2, \ldots, j_t}.$$

Clearly, for arbitrary l and m, similar Plücker relations hold for the left minor determinants of $\overline{V^*}$.

THEOREM 3.3. *Suppose that $l > 0, m > 0$ and set $W = lV + mV^*$. Then the ideal of syzygies for the system $\mathscr{M}$ of polynomials is generated by the Plücker relations for the lower minor determinants of $\bar{V}$ and that for the left minor determinants of $\overline{V^*}$ if and only if $l + m \leqslant n$.*

Proof. To prove the "if" part, it is sufficient to consider the case $l+m=n$. Recall that $\mathscr{M}$ consists of the lower minor determinants of $\bar{V}$, the left minor determinants of $\overline{V^*}$, and the elements of the matrix $C=\overline{V^*}\,\bar{V}$. Let $\sum_\alpha a_\alpha c^\alpha=0$ be a relation among the generators, where c^α is a monomial in the C_i^j-s, a_α is a polynomial in the minor determinants. The assertion of the Theorem amounts to prove that a_α belongs to the ideal of syzygies, for any α. This will be proven if we check for generic fibers $F=\pi_{U,lV}^{-1}(\xi)$, $\xi\in\mathscr{F}_{l,l}$ and $F^*=\pi_{U,mV^*}^{-1}(\eta)$, $\eta\in\mathscr{F}_{m,m}$ that the restrictions of the matrix elements of C to $F\times F^*$ are algebraically independent.

Let us divide the matrices $\bar{V}$ and $\overline{V^*}$ into submatrices,

$$\overline{V^*}=(X\,|\,Y),\qquad \bar{V}=\left(\frac{Z}{W}\right),$$

where X is a $m\times m$-matrix, Y is a $m\times l$-matrix, Z is a $m\times l$-matrix, and W is a $l\times l$-matrix. Fixing matrices X and W, we fix values of all the minor determinants from $\mathscr{M}$. We have $C=XZ+YW$. If $\det(W)\neq 0$ (this is true in general position), then we can choose matrices Y,Z to get any $m\times l$-matrix as C. Thus the "if" part is proven.

The "only if" part. Take l,m such that $1\leqslant l,m\leqslant n$, $l+m>n$ and set $s=l+m-n$, $r=n-l+1$. Denote by a_i^j, b_i^j, c_i^j the element in the ith row and the j-th column of the matrix $\overline{V^*}$, $\bar{V}$, and C, respectively. Denote by $\varepsilon^{a\cdots b}$ and $\varepsilon_{a\cdots b}$ the determinant tensors. In this notation, $\varepsilon^{i_1\cdots i_m}a_{i_1}^1\cdots a_{i_m}^m$ is the left minor determinant of order m of $\overline{V^*}$ and $\varepsilon_{j_1\cdots j_l}b_r^{j_1}\cdots b_n^{j_l}$ is the lower minor determinant of order l of $\bar{V}$. We claim that the following relation holds (for $m=n$ this was indicated to us by E. B. Vinberg):

$$\begin{aligned}&\varepsilon^{i_1\cdots i_m}a_{i_1}^1\cdots a_{i_m}^m\varepsilon_{j_1\cdots j_l}b_r^{j_1}\cdots b_n^{j_l}\\ &\qquad=\frac{1}{s!}\varepsilon^{i_1\cdots i_m}a_{i_1}^1\cdots a_{i_{r-1}}^{r-1}\varepsilon_{j_1\cdots j_l}b_{m+1}^{j_{s+1}}\cdots b_n^{j_l}c_{i_r}^{j_1}\cdots c_{i_m}^{j_s}.\end{aligned}\tag{1}$$

To prove this formula, we rewrite the right hand side, using $c_i^j=a_i^kb_k^j$:

$$\frac{1}{s!}\varepsilon^{i_1\cdots i_m}a_{i_1}^1\cdots a_{i_{r-1}}^{r-1}a_{i_r}^{k_1}\cdots a_{i_m}^{k_s}\varepsilon_{j_1\cdots j_l}b_{k_1}^{j_1}\cdots b_{k_s}^{j_s}b_{m+1}^{j_{s+1}}\cdots b_n^{j_l}.\tag{2}$$

Let $S(k_1,\dots,k_s)$ denote the sum of terms in formula (2) with fixed $k_1,\dots,k_s$. Clearly, if $\{k_1;\cdots;k_s\}\neq\{r;\cdots;m\}$, then $S(k_1,\dots,k_s)=0$. Furthermore, $S(k_1,\dots,k_s)$ is symmetric by $k_1,\dots,k_s$ and for $(k_1,\dots,k_s)=(r,\dots,m)$ $S(k_1,\dots,k_s)$ equals the left hand side of (1).

Therefore the relation (1) holds. Clearly, the right hand side is a polynomial in the left minor determinants of order $m-s$ of $\overline{V^*}$, the lower minor determinants of order $l-s$ of $\bar{V}$, and the matrix elements of C. It is not hard to check that this relation among the generators of $\mathbf{k}[W]^U$ cannot be obtained from relations of smaller degrees. ∎

Now we are able to deduce a proof of Theorem 1.1 for $l+m \leqslant n$:

THEOREM 3.4. *For $W = lV + mV^*$, $l+m \leqslant n$, $\mathbf{k}[W]^U$ is generated by $\mathscr{M}$.*

Proof. Let A be a subalgebra in $\mathbf{k}[W]^U$ generated by the elements of $\mathscr{M}$ and set $X = \operatorname{Spec} A$. Applying consequently Theorems 3.3 and 3.1, we get:

$$X \cong (lV)_U \times (mV^*)_U \times \mathbf{k}^{lm}.$$

Since the subalgebras of covariants are integrally closed, this isomorphism implies that X is a normal affine variety.

Clearly, we may assume $l > 0, m > 0$. Consider linear covariants $f = \bar{V}^1_n$, $g = \overline{V^*}^1_1$. We need to prove $A = \mathbf{k}[W]^U$. We claim local versions of this equality:

$$\mathbf{k}[W]^U_{(f)} = A_{(f)}, \qquad \mathbf{k}[W]^U_{(g)} = A_{(g)}. \tag{3}$$

Then the equality itself follows. Indeed, the equalities (3) imply that any covariant $h \in \mathbf{k}[W]^U$ is a rational function on X. Moreover, this rational function is regular outside the subvariety $X \cap \{f = g = 0\}$. Since this subvariety is of codimension 2 in X and X is normal, we get h is a regular function on X. Thus $\mathbf{k}[W]^U = \mathbf{k}[X] = A$.

So we only need to prove the claim. To check (3), we apply induction on n. The base of induction is the case $n = 2$, where V and V^* are isomorphic as SL_2-modules and the statement $\mathbf{k}[W]^U = A$ follows from Theorem 3.1.

Assume $n > 2$ and $\mathbf{k}[W]^U = A$ is proven for $\dim V < n$. Because of the symmetry of V and V^*, it is sufficient to check $\mathbf{k}[W]^U_{(f)} = A_{(f)}$. We apply the theorem of local structure of Brion–Luna–Vust [BLV].

Denote by $x^j_i = \bar{V}^j_i$ the ith coordinate of the jth vector. Similarly, set $\xi^t_s = \overline{V^*}^t_s$. In particular, $f = x^1_n$, $g = \xi^1_1$. Set $W_f = \{x \in W \mid f(x) \neq 0\}$. Define a mapping:

$$\psi_f\colon W_f \to \mathfrak{gl}(V)^*, \qquad \psi(x)(\xi) = \frac{(\xi f)(x)}{f(x)}.$$

Denote by P_f the stabilizer in $G = GL(V)$ of the line $\langle f \rangle$. Clearly, P_f is a parabolic subgroup in G containing U and ψ_f is P_f-equivariant.

Let $x \in V \subseteq lV \subseteq W$ be the last vector of the chosen basis in the first copy of V. Put $\Sigma = \psi_f^{-1}(\psi_f(x))$. Denote by L the stabilizer of $\psi_f(x)$ in P_f. By [BLV], L is a Levi subgroup of P_f and the natural morphism

$$P_f *_L \Sigma \to W_f, \qquad (p, \sigma) \to p\sigma$$

is a P_f-equivariant isomorphism. Therefore we have:

$$\mathbf{k}[W]^U_{(f)} \cong \mathbf{k}[W_f]^U \cong \mathbf{k}[P_f *_L \Sigma]^U.$$

Also, $P_f = UL$. Hence,

$$\mathbf{k}[P_f *_L \Sigma]^U \cong \mathbf{k}[\Sigma]^{U\cap L} = \mathbf{k}[\Sigma]^{U(L)},$$

where $U(L)$ is a maximal unipotent subgroup in L.

Calculating, we have $L \cong GL_{n-1} \times \mathbf{k}^*$, $\Sigma = (\mathbf{k}^* x) \times ((l-1)\,V + mV^*)$. In other words, Σ is defined by the condition that the first column of $\bar{V}$ is $(0, \ldots, 0, c)^\top$, $c \in \mathbf{k}^*$. Furthermore,

$$\begin{aligned}\mathbf{k}[\Sigma]^{U(L)} \cong \mathbf{k}[x^1_n, \ldots, x^l_n]_{(x^1_n)} &\otimes \mathbf{k}[\xi^n_1, \ldots, \xi^n_m] \\ &\otimes \mathbf{k}[(l-1)\,\mathbf{k}^{n-1} + m\mathbf{k}^{(n-1)*}]^{U(GL_{n-1})}.\end{aligned}$$

The inequality $l-1+m \leqslant n-1$ allows us to apply the induction hypothesis and we get the generators of $\mathbf{k}[(l-1)\,\mathbf{k}^{n-1} + m\mathbf{k}^{(n-1)*}]^{U(GL_{n-1})}$ as certain functions of a $(n-1)\times(l-1)$ matrix $\overline{V_1}$ and a $m \times (n-1)$ matrix $\overline{V^*_1}$, where $\overline{V_1}$ is a submatrix of $\bar{V}$ and $\overline{V^*_1}$ is a submatrix of $\overline{V^*}$. We therefore know all the generators of $\mathbf{k}[\Sigma]^{U(L)}$ as functions on Σ and can easily check that each of them can be obtained as a polynomial in the restrictions of the elements of $\mathscr{M}$ to Σ. Thus $\mathbf{k}[W]^U_{(f)} = A_{(f)}$ and the theorem is proven. ▌

4. A PROOF OF THE MAIN THEOREM FOR $GL(V)$

In this section we deduce from results of the previous one a complete proof of Theorem 1.1 for $GL(V)$, using a general result on covariants of reductive groups, as follows.

Let G be a reductive group, let $U \subseteq G$ be a maximal unipotent subgroup, and let W be a G-module. Put:

$$P = \mathbf{k}[W] \supseteq Q = \mathbf{k}[W]^U \supseteq R = \mathbf{k}[W]^G. \tag{4}$$

Let P_+, Q_+, R_+ be the maximal homogeneous ideals of the corresponding graded algebras. The natural linear map $R_+/R_+^2 \to P_+/P_+R_+$ is injective (this follows, e.g., from 4.1 below). On the other hand, the vector space R_+/R_+^2 can be thought of as the linear span of the generators of R. Using this idea, Schwarz obtained some estimates on the set of generators of R ([Sch87], see 4.3 below). To apply Schwarz's idea for describing generators of Q we need the following lemma:

LEMMA 4.1. *The natural map* $\gamma\colon Q_+/Q_+R_+ \to P_+/P_+R_+$ *is injective.*

Proof. The map γ is well defined, because $Q_+ \subseteq P_+$. We need to prove that a relation $f = \sum_{i=1}^n \varphi_i f_i$, with $f \in Q_+, f_i \in R_+, \varphi_i \in P_+, i = 1, \ldots, n$ implies that a similar relation $f = \sum_{i=1}^m \psi_j g_j$ exists with $g_j \in R_+, \psi_j \in Q_+$, $j = 1, \ldots, m$. Clearly, we may assume that f, φ_i, f_i are homogeneous polynomials and f, φ_i are weight vectors of a maximal torus T of G normalizing U, of the same weight χ. For a dominant weight μ of T denote by $I(\mu)$ the corresponding isotypic component of G-module $\mathbf{k}[W]$, i.e., the sum of all irreducible factors with highest weight μ. Since f is a covariant, a highest weight vector, the weight χ is dominant and $f \in I(\chi)$. Clearly, $I(0)\, I(\mu) \subseteq I(\mu)$; hence, our relation remains true if we replace each f_i by its projection to $I(\chi)$ along the sum of all other isotypic components. So we may assume $f_1, \ldots, f_n \in I(\chi)$. Since χ is the highest weight of $I(\chi)$, we get $f_1, \ldots, f_n \in I(\chi)^U$. This completes the proof. ∎

Now assume that W is endowed with an action of a reductive group H commuting with that of G. Clearly, Q and R are H-stable and we have:

COROLLARY 4.2. *The H-module Q_+/Q_+^2 is isomorphic to a submodule of the H-module P_+/P_+R_+.*

Proof. Since the ideal Q_+R_+ of Q is contained in Q_+^2, the natural map $Q_+/Q_+R_+ \to Q_+/Q_+^2$ is surjective. So by Lemma 4.1, we have:

$$Q_+/Q_+^2 \twoheadleftarrow Q_+/Q_+R_+ \hookrightarrow P_+/P_+R_+.$$

Since the above mappings are H-equivariant, the statement is proven. ∎

In the framework of classical invariant theory, a procedure of *polarization* is frequently used, as follows. For an action $G\colon V \oplus U$, an invariant $f \in S^p(V^*) \otimes S^q(U^*) \subseteq \mathbf{k}[V+U]^G$, and any non-negative integers $p_1, \ldots, p_k$ such that $p_1 + \cdots + p_k = p$, the composition of f with the natural $GL(V)$-equivariant map $S^{p_1}(V^*) \otimes \cdots \otimes S^{p_k}(V^*) \to S^p(V^*)$ is also a G-invariant polynomial, which we call a polarization of f. The polarizations of polarizations of f are also considered as polarizations of f. In such a way, describing f, we obtain a lot of other invariants.

THEOREM 4.3. *Let V be a G-module.*

1. *If the module V is symplectic,* $\dim V = 2r$, *then for $W = lV$ and arbitrary l both algebras* $\mathbf{k}[W]^G$ *and* $\mathbf{k}[W]^U$ *are generated, modulo polarization, by polynomials involving at most r copies of V.*

2. *For $W = lV + mV^*$ and arbitrary l, m both algebras $\mathbf{k}[W]^G$ and $\mathbf{k}[W]^U$ are generated, modulo polarization, by polynomials involving t copies of V and s copies of V^* such that $t + s \leqslant \dim V$.*

Proof. 1. Put $H = GL_l$; H acts naturally on W. Consider a homogeneous generator f of R or Q as an element of the H-module R_+/R_+^2 or Q_+/Q_+^2, respectively. Clearly, we may assume that f belongs to an irreducible H-submodule $E \subseteq R_+/R_+^2$ or $E \subseteq Q_+/Q_+^2$, respectively. Observe that f is a polarization of a polynomial involving p copies of V if and only if the highest weight of E is a sum of fundamental weights φ_i of $H = GL_l$ with $i \leqslant p$. By Corollary 4.2, it is sufficient (for both R and Q) to show that a weight $\chi = k_1\varphi_1 + \cdots + k_l\varphi_l$ is *irrelevant* if $k_t > 0$ for some $t > r$, i.e., that any irreducible H-factor of P_+ with highest weight χ is contained in P_+R_+. Moreover, by [Sch87, 1.20], it is sufficient to check that $\chi = \varphi_{r+1}$ is irrelevant. This is equivalent to an inclusion $\bigwedge^{r+1} V^* \subseteq P_+R_+$ for any subspace $\bigwedge^{r+1} V^* \subseteq \mathbf{k}[(r+1)V]$. By assumption, there exists a G-invariant nondegenerate skew form $\omega \in \bigwedge^2 V^*$. The well known equality $\bigwedge^{r+1} V^* = \omega \wedge \bigwedge^{r-1} V^*$ yields the above inclusion.

2. Put $H = GL_l \times GL_m$; H acts naturally on W. Arguing as above, we reduce the assertion of the theorem to the inclusion $\bigwedge^t V^* \otimes \bigwedge^s V \subseteq P_+R_+$ for any t, s such that $t + s = \dim V + 1$, where $\bigwedge^t V^* \otimes \bigwedge^s V$ is thought of as a subspace in $\mathbf{k}[tV + sV^*]$. Clearly, increasing l or m, if necessary, we may assume $l = m$. Then we have $W = l(V + V^*)$ with G-module $V + V^*$ being symplectic. By the above proof of assertion 1, we have $\bigwedge^{n+1}(V^* + V) \subseteq P_+R_+$, where $n = \dim V = (\dim(V + V^*))/2$. The natural decomposition

$$\bigwedge^{n+1}(V^* + V) = \bigoplus_{t+s=n+1} \bigwedge^t V^* \otimes \bigwedge^s V$$

concludes the proof. ∎

Remark. Both results about $\mathbf{k}[W]^G$ are proven in [Sch87, Theorem 1.22.1 and Remark 1.23.2]. Moreover, the above proof is exactly that from [Sch87]. Actually, thanks to 4.2, Schwarz's proof for R works also for Q. It should be also said that the first assertion follows from results of [We].

Since for $G = GL(V)$ all invariants in $\mathscr{M}$ are multilinear and give no polarizations different from themselves, 3.4 and 4.3 yield:

Corollary 4.4. *For $G = GL(V)$, $W = lV + mV^*$ and arbitrary l, m, the algebra $\mathbf{k}[W]^U$ is generated by $\mathscr{M}$.*

5. WEYL'S CONSTRUCTION AND THE MAIN THEOREM FOR COVARIANTS FOR $O(V)$ AND $Sp(V)$

Let G be one of $O(V)$, $Sp(V)$, $W = lV$. Assume that a G-invariant bilinear form Q on V, a maximal unipotent subgroup $U(G)$, and a maximal torus $T(G)$ are chosen as in Section 2 and write $T = T(G)$, $U = U(G)$.

Examining our systems $\mathscr{M}$ of generators in Section 2, we see that in the cases (B) and (C) all generators of $\mathbf{k}[W]^U$ belong either to $\mathbf{k}[W]^G$ or to $\mathbf{k}[W]^{U(GL)}$ and in the case (D) the same is true for almost all generators.

By Theorem 3.1, $\mathbf{k}[W]^{U(GL)}$ is generated by the lower minor determinants. Therefore Theorem 1.1 for the cases (B), (C) would follow from the following claim: any generator of $\mathbf{k}[W]^U$ different from the scalar products $Q(v_i, v_j)$ is $U(GL)$-invariant. At first sight, an idea to prove the above claim seems to be naive. However, it works. This claim turns out to be connected with a construction for irreducible representations of orthogonal and symplectic groups from the famous book [We] by Weyl. Below we present the results of Weyl and a proof of 1.1. We also need some generalities about tensor representations of $GL(V)$ and symmetric groups. Our general reference for all well known facts mentioned below is a modern book [FH]; we also follow the notation of this book.

For any positive integer d, consider the tensor power $V^{\otimes d}$. Consider the natural action $GL(V)\colon V^{\otimes d}$ and also the action of the symmetric group S_d on $V^{\otimes d}$ by permuting the factors: $\pi(v_1 \otimes \cdots v_d) = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)}$. Clearly, both actions commute with each other and we have an action of $GL(V) \times S_d$ on $V^{\otimes d}$.

Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ be a partition of $d = \lambda_1 + \cdots + \lambda_n$ with at most $n = \dim V$ non-zero parts. To emphasize the relation of λ and d we write $\lambda \vdash d$. Let $S_\lambda V \subseteq V^{\otimes d}$ be the image of the *Young symmetrizer* $c_\lambda\colon V^{\otimes d} \to V^{\otimes d}$; $S_\lambda V$ is an irreducible $GL(V)$-module with highest weight

$$(\lambda_1 - \lambda_2)\,\varphi_1 + \cdots + (\lambda_{n-1} - \lambda_n)\,\varphi_{n-1} + \lambda_n \varphi_n. \tag{5}$$

On the other hand, by V_λ we denote the irreducible S_d-module corresponding to λ. Then we have (e.g., [FH, p.87]):

$$V^{\otimes d} = \bigoplus_{\lambda \vdash d} S_\lambda V \otimes V_\lambda. \tag{6}$$

For each pair $1 \leqslant i < j \leqslant d$, let $\Phi_{ij}\colon V^{\otimes d} \to V^{\otimes d-2}$ be a contraction:

$$\Phi_{ij}(v_1 \otimes \cdots \otimes v_d) = Q(v_i, v_j)\, v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes v_d.$$

Let $V_0^{\otimes d}$ denote the intersection of the kernels of all these contractions. Furthermore, let $\sigma \in V \otimes V$ denote a G-invariant element corresponding to the form Q. Let $Q_+^{\otimes d}$ denote the linear span of all tensors of type $\pi(\sigma \otimes A)$, where $A \in V^{\otimes d-2}$, $\pi \in S_d$. Note that the subspaces $V_0^{\otimes d}, V_+^{\otimes d}$ are $G \times S_d$-stable. The proof of the following lemma is straightforward:

LEMMA 5.1 ([We, 5.6.A], [FH, 17.12]). $V^{\otimes d} = V_0^{\otimes d} \oplus V_+^{\otimes d}$.

Recall that a partition $\lambda \vdash d$ can be depicted by a Young diagram containing d boxes such that the ith row contains λ_i boxes, $i = 1, 2, \ldots$. Now we formulate a part of the results of Weyl about $V^{\otimes d}$.

THEOREM 5.2. *Let λ be a partition of d.*

1. *$S_\lambda V \cap V_0^{\otimes d}$ is non-zero if and only if:*

for $G = Sp(V)$: the length of the first column of the Young diagram of λ is at most $r = (\dim V)/2$.

for $G = O(V)$: the sum of the lengths for the first two columns of the Young diagram of λ is at most $n = \dim V$.

2. *If $S_\lambda V \cap V_0^{\otimes d} \neq \{0\}$, then this is an irreducible G-module.*

Remark. For $O(V)$, the above results appear in [We] as Theorems 5.7.A, 5.7.C, and 5.7.H; for $Sp(V)$, we find them on pp. 174–175 without proofs. In [FH] see Theorems 17.11, 19.19, and 19.22.

We need some more observation concerning Weyl's construction. Let us numerate the factors of the tensor product $V^{\otimes d}$ by the boxes of the diagram of λ. Let $u_\lambda \in V^{\otimes d}$ be a decomposable tensor such that the factor of u_λ corresponding to a box is e_i, if this box is in the ith row of the Young diagram, where e_i is the ith vector of the basis chosen in Section 2. By definition, the vector $v_\lambda = c_\lambda u_\lambda$ is the result of the alternation of u_λ along the columns of the Young diagram. For instance, if the Young diagram consists of exactly one column of length $p \leqslant n$, then $u_\lambda = e_1 \otimes \cdots \otimes e_p$, $v_\lambda = e_1 \wedge \cdots \wedge e_p \in S_\lambda V = \wedge^p V$. Clearly, for any λ, v_λ generates the unique $U(GL)$-invariant line in $S_\lambda V$.

PROPOSITION 5.3. *If λ satisfies the conditions from* 5.2.1, *then $v_\lambda \in V_0^{\otimes d}$.*

Proof. Denote by p the length of the first column of the Young diagram of λ. If $2p \leqslant n$, then the subspace $\langle e_1, \ldots, e_p \rangle \subseteq V$ is isotropic with respect to Q. Hence, $\Phi_{ij}(u_\lambda) = 0$ for any i, j and $u_\lambda \in V_0^{\otimes d}$. Since $V_0^{\otimes d}$ is S_d-stable, we get $v_\lambda \in V_0^{\otimes d}$.

If $2p > n$, then $G = O(V)$ and all other columns of the Young diagram are shorter than $n+1-p$. Therefore a contraction $\Phi_{ij}(u_\lambda)$ is non-zero only if i and j belong to the first column. Hence, for the alternation of u_λ by the first column all contractions are zero, because Q is symmetric. Thus we get $v_\lambda \in V_0^{\otimes d}$. ▮

Corollary 5.4. *Any $GL(V)$-irreducible submodule $L \subseteq V^{\otimes d}$ is a direct sum of its intersections with $V_0^{\otimes d}$ and $V_+^{\otimes d}$ and $L \cap V_0^{\otimes d}$ is either zero or an irreducible G-module containing $L^{U(GL)}$.*

Proof. Since the Young symmetrizer c_λ is a sort of averaging over S_d and because $V_0^{\otimes d}$ and $V_+^{\otimes d}$ are S_d-stable, we have by 5.1:

$$S_\lambda V = c_\lambda V^{\otimes d} = c_\lambda (V_0^{\otimes d} \oplus V_+^{\otimes d}) = (S_\lambda V \cap V_0^{\otimes d}) \oplus (S_\lambda V \cap V_+^{\otimes d}).$$

By (6), there is a partition λ such that $L = S_\lambda V \otimes e \subseteq S_\lambda V \otimes E_\lambda$ for some $e \in E_\lambda$. Thus the above decomposition of $S_\lambda V$ yields that of L. Finally, the properties of the G-module $L \cap V_0^{\otimes d}$ follow from the above presentation of L, 5.2, and 5.3. ▮

Now we are going to explain how Theorem 1.1 for $O(V)$, $Sp(V)$ follows from the above construction.

Corollary 5.5. *For $G = O(V), Sp(V)$, $W = lV$, and arbitrary l, the algebra $\mathbf{k}[W]^U$ is generated by $\mathscr{M}$.*

Proof. Let f be a generator of $\mathbf{k}[W]^U$ such that $f \notin \mathbf{k}[W]^G$, $\deg f = d$. Let us confuse V^* with V by means of the form Q and consider the symmetric algebras of the modules V and W instead of $\mathbf{k}[V]$ and $\mathbf{k}[W]$. Clearly, we may assume f to be sitting in an irreducible $GL(V)$-submodule $E \subseteq S^d(W)$. Furthermore, we regard $S^d(W)$ as the set of all S_d-invariant point in $W^{\otimes d}$ by the natural action of S_d.

On the other hand, we have an isomorphism of $GL(V)$-modules: $W^{\otimes d} \cong (lV)^{\otimes d} \cong l^d V^{\otimes d}$. Hence, the decomposition of $V^{\otimes d}$ from 5.1 yields a decomposition of $W^{\otimes d}$:

$$W^{\otimes d} = W_0^{\otimes d} \oplus W_+^{\otimes d}, W_0^{\otimes d} = l^d V_0^{\otimes d}, W_+^{\otimes d} = l^d V_+^{\otimes d}.$$

One can show that $W_0^{\otimes d}$ and $W_+^{\otimes d}$ are S_d-stable. Then $S^d(W)$ decomposes:

$$S^d(W) = (W^{\otimes d})^{S_d} = (S^d(W) \cap W_0^{\otimes d}) \oplus (S^d(W) \cap W_+^{\otimes d}).$$

In particular, we have $f = f_0 + f_+$, $f_0, f_+ \in S^d(W)$, $f_0 \in W_0^{\otimes d}$, $f_+ \in W_+^{\otimes d}$. Moreover, f_0 and f_+ are U-invariant and T-homogeneous, because $W_0^{\otimes d}$ and $W_+^{\otimes d}$ are G-stable. By definition, the covariant f_+ is a linear combination of the scalar products $Q(v_i, v_j)$ with polynomial coefficients. Hence, 4.1 implies $f_+ \in Q_+ R_+$; i.e., f_+ is a polynomial in covariants of smaller degree. Therefore it is sufficient to prove that f_0 belongs to the algebra generated by the elements of $\mathscr{M}$.

For any positive integer $t \leqslant l^d$, denote by $\pi_t\colon W^{\otimes d} \to V^{\otimes d}$ the projection onto the tth factor of the direct sum $W^{\otimes d} = l^d V^{\otimes d}$. Since this projection is $GL(V)$-equivariant, $\pi_t(E)$ is either $\{0\}$ or an irreducible $GL(V)$-module isomorphic to E. By (6), if the latter is the case, then there exists a partition λ of d such that $\pi_t(E) \cong S_\lambda V$; note that the isomorphism $\pi_t(E) \cong E$ implies that λ does not depend on t.

Fix t and assume $\pi_t(E) \neq \{0\}$. Applying 5.4, we get $\pi_t(f_0) \in \pi_t(E) \cap V_0^{\otimes d}$ and the G-module $X = \pi_t(E) \cap V_0^{\otimes d}$ is irreducible and contains $(\pi_t(E))^{U(GL)}$. Denote by g a generator of the line $(\pi_t(E))^{U(GL)}$; since $U(GL)$ contains U, g is U-invariant. Thus we get $\pi_t(f_0) \in X^U$, $g \in X^U$.

At this point of the proof, we have two cases: with respect to the connected component G^0 of the unity of G X can be either irreducible or reducible. It is well known that in the cases (B), (C) any irreducible G-module is also irreducible with respect to G^0. In the case (D) an irreducible G-module is either irreducible with respect to G^0 or splits into a sum of two irreducible factors. The latter is the case if and only if the highest weights of both factors are conjugate by the order 2 automorphism of the Dynkin diagram D_r (fixing the first simple root for D_4). By 5.4, the highest weight of one of two factors is the restriction of the weight from (5) to T. Calculating this restriction, we get that the G^0-module X is reducible if and only if $G = O(V)$, $n = 2r$ and λ has exactly r parts so that $\lambda_r > 0$, $\lambda_{r+1} = 0$.

Assume that X is irreducible with respect to G^0. Then $\dim X^U = 1$; hence, $\pi_t(f_0)$ is a multiple of g. Therefore $\pi_t(f_0)$ is $U(GL)$-invariant. Furthermore, since for all t such that $\pi_t(f_0) \neq 0$ the partition λ is the same, the projections $\pi_t(f_0)$ are $U(GL)$-invariant for all t. Thus f_0 itself is $U(GL)$-invariant, $f_0 \in \mathbf{k}[W]^{U(GL)}$. Since f is a generator of $k[W]^U$, 3.1 yields f_0 is a lower minor determinant. Moreover, if $G = Sp(V)$ by 5.2.1 the partition λ has at most r parts, i.e., f_0 is a lower minor determinant of order $\leqslant r$ (c.f. 4.3.1).

Assume that X is reducible with respect to $G^0 = SO(V)$. Then X is a sum of two irreducible $SO(V)$-factors. Let $\theta \in O(V) \setminus SO(V)$ be the operator interchanging the rth and the $r+1$th elements of the basis chosen in Section 2 and acting trivially on the other basis elements. Clearly, θ normalizes T and U and interchanges the factors of X. More precisely, the highest weight vectors of the factors are g and $\theta(g)$. Since g and $\theta(g)$ have different weights with respect to T, $\pi_t(f_0)$ is a multiple either of g or of $\theta(g)$. Moreover, the choice is determined by the weight of $\pi_t(f_0)$ and hence, does not depend on t. Arguing as above, we get either $f_0 \in \mathbf{k}[W]^{U(GL)}$ or $\theta(f_0) \in \mathbf{k}[W]^{U(GL)}$. If $f_0 \in \mathbf{k}[W]^{U(GL)}$, then we are done. If $\theta(f_0) \in \mathbf{k}[W]^{U(GL)}$, then $\theta(f_0)$ is a polynomial in the lower minor determinants. That λ contains exactly r parts is equivalent to the maximal order of lower minor determinants involved in $\theta(f_0)$ being exactly r. Clearly, a lower minor determinant of order less than r is θ-invariant and for that of order r the image under θ is the corresponding non-lower minor determinant of order r from the system $\mathscr{M}$ (see Section 2). Therefore $f_0 = \theta(\theta(f_0))$ is a polynomial in elements of the set $\mathscr{M}$. ▮

6. A UNIFORM PROOF OF THE MAIN THEOREM

In the previous sections we gave proofs of Theorem 1.1 for each of the groups $GL(V)$, $O(V)$, $Sp(V)$. Here we present another proof of this result. As a matter of fact, this was the first proof we found. An advantage of this proof is that for all three groups we apply the same idea. As in the proof of 3.1, we consider an action of a reductive Lie algebra on $\mathbf{k}[W]$ commuting with that of G. Applying the joint action of G and this Lie algebra, we reduce 1.1 to a statement that we can check. The construction of such a Lie algebra in terms of differential operators go back to Howe [Ho]. We keep the notation of [Ho] but consider a slightly more general setting.

Let W be a finite-dimensional $\mathbf{k}$-vector space. Denote by

$$\mathfrak{gr} = \mathfrak{gr}_{(2,0)} \oplus \mathfrak{gr}_{(1,1)} \oplus \mathfrak{gr}_{(0,2)} \subseteq \operatorname{End} \mathbf{k}[W]$$

the linear subspace of differential operators with the prescribed by the index degree and order. Namely, $\mathfrak{gr}_{(2,0)}$ are the homogeneous regular functions on W of degree 2 acting on $\mathbf{k}[W]$ by multiplication; $\mathfrak{gr}_{(0,2)}$ are the constant coefficients differential operators of order 2; $\mathfrak{gr}_{(1,1)}$ is nothing but the Lie algebra $\mathfrak{gl}(W)$.

Clearly, $\mathfrak{gr}$ is a Lie subalgebra in $\operatorname{End} \mathbf{k}[W]$, and moreover, $\mathfrak{gr}$ is isomorphic to $\mathfrak{sp}(W \oplus W^*)$, with respect to the natural symplectic form on $W \oplus W^*$.

Assume now that $G \subseteq GL(W)$ is a reductive subgroup. Then G acts on $\mathfrak{gr}$; consider the invariants:

$$\Gamma' = \mathfrak{gr}^G, \qquad \Gamma'_{(2,0)} = \mathfrak{gr}^G_{(2,0)}, \qquad \Gamma'_{(1,1)} = \mathfrak{gr}^G_{(1,1)}, \qquad \Gamma'_{(0,2)} = \mathfrak{gr}^G_{(0,2)}.$$

Clearly, $\Gamma' = \Gamma'_{(2,0)} \oplus \Gamma'_{(1,1)} \oplus \Gamma'_{(0,2)}$ is also a Lie subalgebra in $\operatorname{End} \mathbf{k}[W]$.

Let I be an isotypic component of the G-module $\mathbf{k}[W]$, that is, the sum of all irreducible factors isomorphic to a given one. Clearly, I is stable under the action of Γ'.

THEOREM 6.1 ([Ho, Theorem 8]). *Assume that the algebra $\mathbf{k}[W \oplus W^*]^G$ of invariants is generated by elements of degree 2. Then I is an irreducible joint (G, Γ')-module.*

Remark. By the first fundamental theorem for the classical groups, the assumption of Theorem 6.1 holds for the pairs (G, W) from 1.1. For these particular cases the above theorem is (a part of) Theorem 8 of [Ho]. However, one can see that the proof in [Ho] works whenever the assumption of Theorem 6.1 holds.

Note that for the classical (G, W) we have $\Gamma'_{(1,1)} = \mathfrak{gl}_l \oplus \mathfrak{gl}_m$,

$$\begin{aligned} \Gamma' &\cong \mathfrak{gl}_{l+m}, && \text{if} \quad G = GL(V), \\ \Gamma' &\cong \mathfrak{sp}_{2l}, && \text{if} \quad G = O(V), \\ \Gamma' &\cong \mathfrak{o}_{2l}, && \text{if} \quad G = Sp(V). \end{aligned}$$

We now show that Theorem 6.1 reduces Theorem 1.1 to a simpler statement. The below reasoning is an analog of that from the proof of Theorem 9 in [Ho].

Clearly, I is a homogeneous submodule of $\mathbf{k}[W]$; denote by I^{min} the subspace of the elements of I of minimal degree. Let $A \subseteq \mathbf{k}[W]^U$ be the subalgebra generated by $\mathscr{M}$. Let $Z \subseteq \mathbf{k}[W]$ be the G-submodule generated by A. Then Theorem 1.1 can be reformulated as $Z = \mathbf{k}[W]$. Assume that $X = Z \cap I^{min}$ is non-zero.

Since the system $\mathscr{M}$ of generators of A is symmetric with respect to permutations of isomorphic factors of G-module W, A is $GL_l \times GL_m$-stable, i.e., $\Gamma'_{(1,1)}$-stable. Hence, Z and X are stable with respect to both G and $\Gamma'_{(1,1)}$.

Let R, $R_{(2,0)}$ etc. be the subalgebras in End $\mathbf{k}[W]$ generated by Γ', $\Gamma'_{(2,0)}$ etc. Consider R as a representation of the universal enveloping algebra of Γ'. Using the Poincaré–Birkhoff–Witt theorem, we obtain

$$R = R_{(2,0)} R_{(1,1)} R_{(0,2)}. \tag{7}$$

Differentiating a polynomial, we decrease its degree; hence, $\Gamma'_{(0,2)} I^{min} = 0$. Therefore $R_{(0,2)} X = X$. Moreover, since X is $\Gamma'_{(1,1)}$-stable, we have by (7), $RX = R_{(2,0)} X = \mathbf{k}[W]^G X$. On the other hand, RX is a non-zero joint (G, Γ')-submodule of I. By Theorem 6.1, $I = RX = \mathbf{k}[W]^G X \subseteq Z$.

Thus we only need to check for any isotypic component I:

$$A \cap I^{min} \neq \{0\}. \tag{8}$$

Note that we may and we will assume $l, m \geqslant n$ for $G = GL(V)$, and $l \geqslant n$, $m = 0$ for $G = O(V)$, $Sp(V)$. Denote by P the set of highest weights of irreducible factors for G^0-module $\mathbf{k}[W]$, where G^0 stands for the connected component of the unity of G. For a graded algebra B and $t \in \mathbf{N}$, denote by B_t the subspace of the elements of degree t. For any $\chi \in P$ set:

$R(\chi)$ is the irreducible representation of G^0 with highest weight χ,
I_χ is the $R(\chi)$-isotypic component of G^0-module $\mathbf{k}[W]$,
$m(\chi) = \min\{t \mid \mathbf{k}[W]_t \cap I_\chi \neq 0\}$,
$n(\chi) = \min\{t \mid A_t \cap I_\chi \neq 0\}$.

By definition, $n(\chi) \geqslant m(\chi)$. For $G = GL(V), Sp(V)$ the condition (8) is equivalent to $n(\chi) = m(\chi)$ for any $\chi \in P$. The function $n(\chi)$ can be computed. Indeed, by definition, $n(\chi)$ is the minimum of degree of the monomials of weight χ in the elements of $\mathscr{M}$. Clearly, we should not involve the G-invariants in a monomial of minimal degree. Then for $G = Sp(V), O(V)$ we do not have much choice for such a monomial and we can write down formulae for $n(\chi)$ as follows. Let $\chi = k_1\varphi_1 + \cdots + k_r\varphi_r$.

For $G = Sp(V)$, we have $n(\chi) = k_1 + 2k_2 + \cdots + rk_r$.

For $G = O(V)$, $n = 2r+1$, k_r is even for $\chi \in P$, and we have:

$$n(\chi) = k_1 + 2k_2 + \cdots + (r-1)\, k_{r-1} + r\frac{k_r}{2}. \tag{9}$$

For $G = O(V)$, $n = 2r$, $k_{r-1} + k_r$ is even for $\chi \in P$, and we have:

$$n(\chi) = k_1 + 2k_2 + \cdots + (r-2)\, k_{r-2} + r\frac{k_{r-1} + k_r}{2} - \min(k_r, k_{r-1}). \tag{10}$$

For $G = GL(V)$, it is difficult to write down $n(\chi)$ explicitly. However, there is a natural algorithm for calculating $n(\chi)$. This algorithm and the above formulae show that $n(\chi)$ is a homogeneous function:

Lemma 6.2. *For any $\chi \in P$, $c \in \mathbf{N}$ we have $n(c\chi) = cn(\chi)$.*

Denote by $\mathfrak{t}$ the Lie algebra of $T(G)$. Let $\mathscr{C} \subseteq \mathfrak{t}^*$ be the Weyl chamber corresponding to $U(G)$. Consider the set

$$\Delta = \left\{ \frac{\chi^*}{t} \mid I(\chi) \cap \mathbf{k}[W]_t \neq 0 \right\} \subseteq \mathscr{C},$$

where χ^* denotes the highest weight of the G^0-module dual to that with highest weight χ. It should be noted that by [Br87], if $\mathbf{k}$ is the field $\mathbf{C}$ of complex numbers, then Δ is the set of rational points in the momentum polytope for the action of the maximal compact subgroup $K \subseteq G^0$ on the projective space $\mathbf{P}(W)$. We do not use this description of Δ in the proof. For the subalgebra $A \subseteq \mathbf{k}[W]$ we introduce an analog of Δ:

$$\tilde{\Delta} = \left\{ \frac{\chi^*}{t} \mid I(\chi) \cap Z_t \neq 0 \right\} \subseteq \Delta.$$

Clearly, if $(t_i, \chi_i), i = 1, \ldots, M$ is the set of degrees and weights of the elements from $\mathscr{M}$ (see Section 2), then

$$\tilde{\Delta} = \mathrm{conv}_{\mathbf{Q}}\left(\frac{\chi_1^*}{t_1}, \ldots, \frac{\chi_M^*}{t_M}\right).$$

Let now $\Phi \subseteq \mathfrak{t}^*$ be the convex hull over the rational numbers of the weights for the action $T(G) : W$. The following statement is an easy exercise on the convex geometry:

LEMMA 6.3. $\tilde{\Delta} \supseteq \Phi \cap \mathscr{C}$.

By definition, we have $\Delta \subseteq \Phi \cap \mathscr{C}$. Therefore $\Delta = \tilde{\Delta} = \Phi \cap \mathscr{C}$.

Suppose that $\mathbf{k}[W]^{U(G)}$ contains an element of degree t and weight χ. Then by definition, $\frac{\chi^*}{t} \in \Delta$. Hence, the equality $\Delta = \tilde{\Delta}$ implies that for some $c \in \mathbf{N}$ there exists an element of A of degree ct and weight $c\chi$. Thus $ct \geqslant n(c\chi) = cn(\chi)$ and $t \geqslant n(\chi)$. In other words, $m(\chi) \geqslant n(\chi)$; hence $m(\chi) = n(\chi)$. This completes the proof of the Theorem for $G = GL(V), Sp(V)$.

Let G be $O(V)$. Consider an irreducible representation ρ of $O(V)$ and its restriction ρ' to $SO(V)$. Here two cases occur:

- either ρ' is also irreducible, $\rho' = R(\chi)$ for some $\chi \in P$
- or else $n = 2r$, $\rho' = R(\chi) + R(\tau(\chi))$, where τ is the automorphism of the Dynkin diagram D_r interchanging the $(r-1)$-th and the r-th roots.

The latter case is more simple: elements of minimal degree in the ρ-isotypic component are the elements of minimal degree in both $I(\chi)$ and $I(\tau(\chi))$ (clearly, $n(\chi) = n(\tau(\chi))$ and $m(\chi) = m(\tau(\chi))$). Hence, the above equality $n(\chi) = m(\chi)$ implies the assertion for such an isotypic component.

Now consider the former case and assume that χ is a dominant weight such that $\tau(\chi) = \chi$ if $n = 2r$. Here for any $\rho' = R(\chi)$ there exist two possibilities for ρ, $R(\chi_+)$ and $R(\chi_-) = R(\chi_+) \otimes \det$, where det is the unique nontrivial character of $O(V)$. Moreover, we define explicitly $R(\chi_+)$ and $R(\chi_-)$ as follows. Let θ be $-Id \in O(V)$ for n odd, and let $\theta \in O(V)$ be the operator defined at the end of the proof of 5.5 for n even. Note that in both cases θ normalizes $T(O)$ and $\theta(\chi) = \chi$. Now we define $R(\chi_\pm)$ by the condition

$$R(\chi_\pm)(\theta)(u_\chi) = \pm u_\chi$$

for the highest weight vector $u_\chi \in R(\chi)$. For instance, if n is even, $2k < n$, then lower minor determinants of order k of $\bar{V}$ generate $R((\varphi_k)_+)$ and those of order $n-k$ generate $R((\varphi_k)_-)$. If n is odd, $2k < n$, then lower minor determinants of $\bar{V}$ of order k generate $R((\varphi_k)_{(k)})$ and those of order $n-k$ generate $R((\varphi_k)_{(n-k)})$, where $(q)=(-1)^q$. Moreover, multiplying the highest weight vectors of two factors of $\mathbf{k}[W]$, we add their weights and multiply their $\pm$ subscripts. Thus we control the structure of the $O(V)$-module Z.

Define $m(\chi_\pm)$, $n(\chi_\pm)$ as above. Then the condition (8) is equivalent to the equality $m(\chi_\pm)=n(\chi_\pm)$ for any χ.

Let B be a monomial in the elements of $\mathscr{M}$ generating the $SO(V)$-module $R(\chi)$ and having the minimal possible degree $n(\chi)$. Clearly, the maximal order $t(\chi)$ of minor determinants involved in B is less than $n/2$. Therefore for n even B generates the $O(V)$-module $R(\chi_+)$, hence $n(\chi_+)=n(\chi)$. For n odd B generates the $O(V)$-module $R(\chi_{(|\chi|)})$, where for $\chi=k_1\varphi_1+\cdots+k_r\varphi_r$, $|\chi|=k_1+\cdots+k_{r-1}+k_r/2$ is the number of minor determinants involved in B. So we have: $n(\chi_{(|\chi|)})=n(\chi)$.

Let B generate $R(\chi_s)$, where $s=\pm$. It is clear that to obtain a monomial of minimal degree in the elements of $\mathscr{M}$ generating $R(\chi_{-s})$ we can replace in B a minor determinant of order $t(\chi)$ by that of order $n-t(\chi)$. Thus we have:

$$n=2r: n(\chi_+)=n(\chi),\ n(\chi_-)=n(\chi)+n-2t(\chi), \tag{11}$$

$$n=2r+1: n(\chi_{(|\chi|)})=n(\chi),\ n(\chi_{-(|\chi|)})=n(\chi)+n-2t(\chi). \tag{12}$$

Assume $n(\chi_s)=n(\chi)$, where $s=\pm$. Then the equality $n(\chi)=m(\chi)$ and the inequality $m(\chi_s)\geqslant m(\chi)$ yield $n(\chi_s)=m(\chi_s)$. However to obtain $n(\chi_{-s})=m(\chi_{-s})$, we need some additional arguments.

Applying induction on n and the theorem of local structure from [BLV] as in the proof of 3.4, we get a local version of the assertion of the theorem. For $f=x_n^1\in\mathbf{k}[V]^{U(O)}$ being the highest weight vector of the dual to the first copy of V we have:

$$\mathbf{k}[W]_{(f)}^{U(O)}=A_{(f)}. \tag{13}$$

Note also that f generate $R((\varphi_1)_{(n)})$. Let g be a highest weight vector of a factor $R(\chi_s)\subseteq\mathbf{k}[W]$ and assume $n(\chi_s)\neq n(\chi)$ (otherwise we are done). Then by (13), for some even j we have $f^jg\in A$. Since f^jg generates $R((\chi+j\varphi_1)_s)$, we have $\deg g+j\geqslant n((\chi+j\varphi_1)_s)$. Using formulae (9), (10), (11), (12), and an easy observation $t(\chi+j\varphi_1)=t(\chi)$, we get $n((\chi+j\varphi_1)_s)=n(\chi_s)+j$. Then we have $\deg g\geqslant n(\chi_s)$. Since g is arbitrary, this inequality implies $m(\chi_s)\geqslant n(\chi_s)$, which is equivalent to the equality $m(\chi_s)=n(\chi_s)$. This completes the proof of the theorem.

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